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# Conservation Laws and Symmetries in Competitive Systems

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## Abstract

We investigate a conservation law of a system of symmetric  $2n$ -dimensional nonlinear differential equations. We use Lagrangian approach and Noether's theorem to analyze Lotka-Volterra type of competitive system. We observe that the coefficients of the  $2n$ -dimensional nonlinear differential equations are strictly restricted when the system has a conserved quantity, and the relation between a conserved system and Lyapunov function is shown in terms of Noether's theorem. We find that a system of the  $2n$ -dimensional first-order nonlinear differential equations in a symmetric form should appear in a binary-coupled form (BCF), and a BCF has a conserved quantity if parameters satisfy certain conditions. The conservation law manifests characteristic properties of a system of nonlinear differential equations and can be employed to check the accuracy of numerical solutions in the BCF.

## 1 Introduction

Nonlinear dynamical systems characterized by self-interactions, self-organizations, spontaneous emergence of order [1], dissipative structure [2] and nonlinear cooperative phenomena have shown essential roles in natural sciences, as well as economy, ecology, and environmental sciences [3, 4]. Nonlinear dynamical systems are difficult to handle unlike linear dynamical systems because their complex interactions and structures make it so hard to understand a response of a system, which may exhibit no simple laws or orders. However, in terms of natural sciences, conservation laws, symmetries and orders in nonlinear dynamical systems are expected to exist even in biology [5], ecology, and economy. The important examples in the field of ecology are those of Malthus (1959) for a population analysis, Alfred Lotka (1925) [6] and Vito Volterra (1926) [9] for predator-prey differential equations known as Lotka-Volterra (LV) equation. Also, in the field of economy, a conserved quantity in the system of business cycle [7] is studied and this business cycle model is regarded as a predator-prey type competitive system, and also a mathematical model for Lanchester Strategic Management is known as a predator-prey type competitive system [8]. We employed the Lagrangian approach to examine a system of  $2n$ -dimensional nonlinear differential equations which contains linear interactions and Lotka-Volterra type of nonlinear interactions.

The research on symmetries and the first integrals of the LV system attracted attention in the 20th century and Lagrangian approach to equations of motion is known as a remarkable method to analyze conservation laws in nonlinear dynamics [10] and generalized to space-time 4-dimensional

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and multi-dimensional systems [11, 12]. The concept of symmetry provides us with conservation laws and enables us to find stable solutions of nonlinear differential equations [14]. Several methods such as the Lie method [15, 16], Painlevé analysis [17], have been used in order to search conservation laws or symmetries. For instance, José Fernández-Núñez studied symmetries of two-dimensional LV system [13]. They discussed a Lagrangian structure in LV system with Lagrangian linear in velocities. However, we found a conservation law which is velocity-independent, and the symmetric nonlinear differential equations of  $2n$ -dimensional ND system will be derived from conservation laws by Noether's theorem [18]. The symmetry and conservation law in the  $2n$ -dimensional first-order differential equations require a binary coupled form, and hence, the system of the nonlinear differential equations becomes the even dimensions  $(2, 4, 6, \dots, 2n)$ . We denote the requirement of the symmetric form as the binary-coupled form (BCF).

In this paper, we discuss the existence of conservation laws in a  $2n$ -dimensional competitive system with general nonlinear interactions. Competitive systems are well known in ecology and biology, as well as in the fields of engineering and information systems [25]. We discuss that symmetries, conservation laws of nonlinear interactions are important in nature by analyzing a general nonlinear dynamical (ND) system using Noether's theorem.

In section 2, we introduce notations, Euler-Lagrange equation, and Noether's theorem to define conservation laws. In section 3, we derive Lagrangian and conservation laws of a general competitive system for the  $2n$ -dimensional ND system. Then we discuss that a symmetric and nonlinear dynamical system should exist in a form of  $2n$ -dimensional nonlinear differential equations when velocity-independent conservation law exists. In section 4, we will illustrate examples of conservation laws in two or three variables in order to explain the results discussed in section 3. In section 5, we shall discuss our results.

## 2 Noether's theorem and conserved quantities

The general formulation of necessary condition for extrema in Lagrangian formulation,  $\mathcal{L}(t, x^k(t), \dot{x}^k(t))$ , is given by

$$\delta J = \delta \int \mathcal{L}(t, x^k(t), \dot{x}^k(t)) dt = 0 \quad (k = 1, \dots, n), \quad (2.1)$$

and all functions,  $x(t) = (x^1(t), \dots, x^n(t))$ ,  $t \in [a, b]$ , belong to  $C^2[a, b]$ , which denotes the set of all continuous functions on the interval  $[a, b]$  and the second derivatives of all functions are continuous. If  $x(t)$  is a relative minimum of the functional  $J$ , the condition (2.1) generates,

$$E_k \equiv \frac{\partial \mathcal{L}}{\partial x^k} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^k} \right) = 0. \quad (2.2)$$

This is the Euler-Lagrange equation which determines equations of motion of a system.

Noether's theorem describes that the Lagrangian with Euler-Lagrange equation is invariant under certain space-time transformations, and invariances of Lagrangian generate respective conservation laws. For example, the time-translation invariance of Lagrangian corresponds to the conservation of energy of a system. Let us consider  $r$ -parameter transformations in general that will be regarded as transformations of configuration space,  $(t, x^1, \dots, x^n)$ -space, depending upon  $r$  real, independent parameters  $\varepsilon^1, \dots, \varepsilon^r$ . The transformations are defined by

$$\begin{aligned} \bar{t} &= t + \tau_s(t, x)\varepsilon^s + o(\varepsilon), \\ \bar{x}^k &= x^k + \xi_s^k(t, x)\varepsilon^s + o(\varepsilon), \end{aligned} \quad (2.3)$$

where  $s$  ranges over  $1, \dots, r$ , and  $o(\varepsilon)$  denotes the terms which go to zero faster than  $|\varepsilon|$ ,  $\lim_{|\varepsilon| \rightarrow 0} o(\varepsilon)/|\varepsilon| = 0$ . The functions  $\tau_s(t, x)$  and  $\xi_s^k(t, x)$  for linear parts of  $\bar{t}$  and  $\bar{x}^k$  with respect to  $\varepsilon$  are commonly called the infinitesimal generators of the transformations. Classical theorem of Emmy Noether on invariant variational problems can be derived under the hypotheses of extrema (2.1), and transformation (2.3) [19]. The result is the  $r$  identities produced by the transformation (2.3),

$$-E_k(\xi_s^k - \dot{x}^k \tau_s) = \frac{d}{dt} \left( \left( \mathcal{L} - \dot{x}^k \frac{\partial \mathcal{L}}{\partial \dot{x}^k} \right) \tau_s + \frac{\partial \mathcal{L}}{\partial \dot{x}^k} \xi_s^k - \Phi_s \right), \quad (2.4)$$

where  $k = 1, \dots, n$  is summed, and  $\Phi_s$  is an arbitrary function defined as a gauge function of the transformation. Note that the arbitrary choice of a gauge function will not change equations of motion of a system, which can be usually used for a convenient expression of conservation laws. If the fundamental integral (2.1) is divergence-invariant under the  $r$  parameter group of transformation (2.3), and if  $E_k = 0$  for  $k = 1, \dots, n$ , then following  $r$  expressions are obtained:

$$\Psi_s \equiv \left( \mathcal{L} - \dot{x}^k \frac{\partial \mathcal{L}}{\partial \dot{x}^k} \right) \tau_s + \frac{\partial \mathcal{L}}{\partial \dot{x}^k} \xi_s^k - \Phi_s = \text{constant}. \quad (2.5)$$

This defines the conserved quantities of a system. Since the expressions  $\Psi$  defined in (2.5) are constant with the condition:  $E_k = 0$  ( $k = 1, \dots, n$ ), they are the first integrals of the differential equations of motion, that is, the conserved quantities. In physical applications, the first integral (2.5) is interpreted as the energy of the system, whose governing equations are  $E_k = 0$ . In general, the expressions  $\Psi$  is constant with respect to time and along any extremal solutions of a system. By employing the formalism, the conservation law and symmetry of nonlinear dynamical differential equations are discussed in the section 3, and we will show the first integrals, the symmetry of  $\Psi$  and its solutions in an explicit coupled two-variables system in section 4, as an example for the current approach.

### 3 Conserved quantities and symmetric BCF in $2n$ -dimensional nonlinear dynamical (ND) systems

Nonlinear dynamical systems are well known in feed-back systems, such as biology, environmental sciences and computer systems, and they are important to understand many complicated interactive structures. Consider a nonlinear dynamical system that has  $2n$  variables. The system has self-interactions, mixing interactions of quadratic forms of all combinations, such as  $x_1^2, x_2^2, \dots, x_1 x_2, x_3 x_4, \dots$ . In this section, we will show the following. (1) We begin to examine a nonlinear dynamical system of  $2n$  variables in symmetric form. The variables  $(x_1, x_2, x_3, \dots, x_{2n})$  are, for example,  $2n$  species of competing creatures in LV system [4], or a cell-structured organism which cooperates together [5]. We will write down a quadratically interacting system as general as possible and investigate the properties of nonlinear interactions as a whole. (2) We will discuss the solution of the  $2n$ -dimensional ND differential system with respect to the conservation law and Noether's theorem by calculating  $\Psi$  explicitly. (3) We will show that a coupled nonlinear dynamical system in BCF has a conserved quantity, or Hamiltonian of the BCF, which is velocity-independent.

The  $2n$ -dimensional ND system in BCF having  $2n$  variables  $(x_1, \dots, x_{2n})$  is generally de-

scribed as,

$$\begin{aligned}\dot{x}_{2k-1} &= \sum_{i=1}^n (a_{2k-1,2i-1}x_{2i-1} + a_{2k-1,2i}x_{2i}) + \sum_{j=1}^{2n} a_{2k-1,2n+j}x_jx_{2k-1} + \sum_{i=1}^{n'} a_{2k-1,4n+i}x_{2i-1}x_{2i}, \\ \dot{x}_{2k} &= \sum_{i=1}^n (a_{2k,2i-1}x_{2i-1} + a_{2k,2i}x_{2i}) + \sum_{j=1}^{2n} a_{2k,2n+j}x_jx_{2k} + \sum_{i=1}^{n'} a_{2k,4n+i}x_{2i-1}x_{2i},\end{aligned}\quad (3.1)$$

where  $k = 1, 2, \dots, n$ . The first summation expresses the linear part of interaction which contains all variables  $x_1, \dots, x_{2n}$ . The second term is the competitive interaction expressed by coupled two variables,  $x_kx_1, \dots, x_kx_{2n}$ . This term is typical in classical LV equations. The third term with prime,  $\sum_{i=1}^{n'} x_{2i-1}x_{2i}$ , expresses all the mixing interactions other than  $i = k$ .

The Lagrangian of (3.1) is given by the following form,

$$\begin{aligned}\mathcal{L} &= \sum_{i=1}^n (\alpha_{2i-1}\dot{x}_{2i-1}x_{2i} + \alpha_{2i}x_{2i-1}\dot{x}_{2i}) + \sum_{i=1}^n \sum_{j=1}^{2n} \alpha_{\{2n+2n(i-1)+j\}}x_{2i-1}x_j \\ &+ \sum_{i=1}^n \sum_{j=1}^{2n} \alpha_{\{2n^2+2n+2n(i-1)+j\}}x_{2i}x_j + \sum_{i=1}^n \sum_{j=1}^{2n} \alpha_{\{4n^2+2n+2n(i-1)+j\}}x_{2i-1}x_{2i}x_j.\end{aligned}\quad (3.2)$$

Applying Euler-Lagrange equation (2.2) to Eq. (3.2), we can get ordinary differential equations; the ordinary differential equation for  $\dot{x}_{2k-1}$  is derived as

$$\begin{aligned}d_{2k,2k-1}\dot{x}_{2k-1} &= \sum_{i=1}^n (\alpha_{2n+2n(i-1)+2k} + \alpha_{2n^2+2n+2n(k-1)+2i-1})x_{2i-1} \\ &+ \sum_{i=1}^n (\alpha_{2n^2+2n+2n(i-1)+2k} + \alpha_{2n^2+2n+2n(k-1)+2i})x_{2i} \\ &+ \sum_{j=1}^{2n} \alpha_{4n^2+2n+2n(k-1)+j}x_jx_{2k-1} + \sum_{i=1}^n \alpha_{4n^2+2n+2n(i-1)+2k}x_{2i-1}x_{2i},\end{aligned}\quad (3.3)$$

and the ordinary differential equation for  $\dot{x}_{2k}$  is derived as

$$\begin{aligned}d_{2k-1,2k}\dot{x}_{2k} &= \sum_{i=1}^n (\alpha_{2n+2n(i-1)+2k-1} + \alpha_{2n+2n(k-1)+2i-1})x_{2i-1} \\ &+ \sum_{i=1}^n (\alpha_{2n^2+2n+2n(i-1)+2k-1} + \alpha_{2n^2+2n+2n(k-1)+2i})x_{2i} \\ &+ \sum_{j=1}^{2n} \alpha_{4n^2+2n+2n(k-1)+j}x_jx_{2k} + \sum_{i=1}^n \alpha_{4n^2+2n+2n(i-1)+2k-1}x_{2i-1}x_{2i},\end{aligned}\quad (3.4)$$

where  $d_{2k,2k-1} = \alpha_{2k} - \alpha_{2k-1}$  and  $d_{2k-1,2k} = -d_{2k,2k-1}$  for all  $k$ . The conditions of parameters can be obtained by comparing (3.1) and (3.3) when the coupled ND differential equations have a

conserved quantity. The conditions of coefficients for  $\dot{x}_{2k-1}$  are given by,

$$\begin{aligned} a_{2k-1,2i-1} &= \frac{1}{d_{2k,2k-1}} (\alpha_{2n+2n(i-1)+2k} + \alpha_{2n^2+2n+2n(k-1)+2i-1}), \\ a_{2k-1,2i} &= \frac{1}{d_{2k,2k-1}} (\alpha_{2n^2+2n+2n(i-1)+2k} + \alpha_{2n^2+2n+2n(k-1)+2i}), \\ a_{2k-1,2n+j} &= \begin{cases} \frac{2}{d_{2k,2k-1}} \alpha_{4n^2+2n+2n(k-1)+2k} & (j = 2k), \\ \frac{1}{d_{2k,2k-1}} \alpha_{4n^2+2n+2n(k-1)+j} & (j \neq 2k), \end{cases} \\ a_{2k-1,4n+i} &= \frac{1}{d_{2k,2k-1}} \alpha_{4n^2+2n+2n(i-1)+2k} \quad (i \neq k), \end{aligned} \quad (3.5)$$

and also by comparing (3.1), (3.4), we get conditions of parameters for  $\dot{x}_{2k}$  ND differential equations,

$$\begin{aligned} a_{2k,2i-1} &= \frac{1}{d_{2k-1,2k}} (\alpha_{2n+2n(i-1)+2k-1} + \alpha_{2n+2n(k-1)+2i-1}), \\ a_{2k,2i} &= \frac{1}{d_{2k-1,2k}} (\alpha_{2n^2+2n+2n(i-1)+2k-1} + \alpha_{2n^2+2n+2n(k-1)+2i}), \\ a_{2k,2n+j} &= \begin{cases} \frac{2}{d_{2k-1,2k}} \alpha_{4n^2+2n+2n(k-1)+2k} & (j = 2k-1), \\ \frac{1}{d_{2k-1,2k}} \alpha_{4n^2+2n+2n(k-1)+j} & (j \neq 2k-1), \end{cases} \\ a_{2k,4n+i} &= \frac{1}{d_{2k-1,2k}} \alpha_{4n^2+2n+2n(i-1)+2k-1} \quad (i \neq k). \end{aligned} \quad (3.6)$$

One should note that if the parameters satisfy these conditions, then the ND differential equations have a conserved quantity. The conserved quantity for time transformation for the coupled ND equations is given by,

$$\begin{aligned} \Psi \equiv & \sum_{i=1}^n \sum_{j=1}^{2n} \alpha_{\{2n+2n(i-1)+j\}} x_{2i-1} x_j + \sum_{i=1}^n \sum_{j=1}^{2n} \alpha_{\{2n^2+2n+2n(i-1)+j\}} x_{2i} x_j \\ & + \sum_{i=1}^n \sum_{j=1}^{2n} \alpha_{\{4n^2+2n+2n(i-1)+j\}} x_{2i-1} x_{2i} x_j. \end{aligned} \quad (3.7)$$

The conserved quantity,  $\Psi(x_1, x_2, \dots, x_{2n})$ , that produces the first order coupled nonlinear differential equations in BCF is conserved and constant with respect to time.

In this ND system, we define the stable solutions as follows. When one substitutes the solutions  $(x_1, x_2, \dots, x_{2n})$  into  $\Psi(x_1, x_2, \dots, x_{2n})$  obtained in certain time range and  $\Psi$  becomes strictly constant, we say that the solutions are stable in the time range. If  $\Psi$  is not constant, solutions are unstable or may not exist. In addition, one can conclude that the coefficients of the ND system are strictly constrained by the conservation law. This is also one of the important results of the conserved ND system. When the coefficients satisfy the relations (3.5) and (3.6), the ND differential equations have solutions that maintain the conserved quantity  $\Psi(x_1, x_2, \dots, x_{2n})$ . If the ND system does not meet conditions (3.5) and (3.6),  $\Psi$  may not be constant with respect to time. It will be shown explicitly in examples in the section 4.

However, because nonlinear coefficients can be assumed to take any values although they are restricted by some conditions, there exist cases that the general nonlinear dynamical system does not have solutions despite the fact that it has formally the conserved quantity  $\Psi$ . For this reason, we mainly concern ourselves to analyze cases in our BCF competitive system which has converged solutions and a conserved quantity  $\Psi$ .

The conserved quantity  $\Psi$  is invariant under exchanges of any variables  $x_i \leftrightarrow x_j$ ;

$$\Psi(x_1, \dots, x_i, \dots, x_j, \dots, x_{2n}) = \Psi'(x_1, \dots, x_j, \dots, x_i, \dots, x_{2n}), \quad (3.8)$$

where  $\Psi'(x_1, \dots, x_j, \dots, x_i, \dots, x_{2n})$  is obtained from  $\Psi$  by renaming of coefficients. Because of the symmetry, the addition of other BCF, for example,  $\Psi(x_{2n+1}, x_{2n+2})$  assuming the similar nonlinear interaction to the whole system, changes only  $n \rightarrow n + 1$  in (3.7), which means the conservation law of the system is unchanged. It indicates that the property of ND system may be kept unchanged even when the nonlinear differential equations are changed to as  $n \rightarrow n + 1$ ;  $\Psi(x_1, \dots, x_{2n}) = \text{constant}$  can be maintained as  $\Psi(x_1, \dots, x_{2n}, x_{2n+1}, x_{2n+2}) = \text{constant}$ , if the coefficients of nonlinear interactions of  $\Psi(x_{2n+1}, x_{2n+2})$  with others are not extremely different. The condition (3.8) could be interpreted in terms of conservation law to maintain stability, homeostasis of a biological system.

The conserved quantity  $\Psi(x_1, \dots, x_{2n}) = \text{constant}$ , can be used to check the accuracy of numerical solutions to the  $2n$  differential equations. Let us suppose that solutions to (3.1),  $x_1, \dots, x_{2n}$  are obtained. Then, one should substitute all solutions to (3.7) to check if  $\Psi$  becomes constant or not. In the Fig. 3.1 and Fig. 3.2, the solutions to classical LV equation in the example of the section 4.2 are shown. The parameters of solution 1 and solution 2 are  $\alpha_1 = -0.02$ ,  $\alpha_2 = -0.02$ ,  $\alpha_3 = 50.0$ ,  $\alpha_4 = 100.0$ ,  $\alpha_5 = 1.0$ , and initial values of solution 1 and solution 2 are  $x_1 = 10.0$ ,  $x_2 = 10.0$  (see (4.12)). The size of interval  $\Delta H$  in Runge-Kutta methods of solution 1 is  $\Delta H = 0.01$ , and solution 2 is  $\Delta H = 0.5$ .

If  $\Psi$  is exactly constant in the assumed interval  $t \in [a, b]$ , the solutions,  $x_1, \dots, x_{2n}$ , are expected to be exact. If  $\Psi \approx \text{constant}$ ,  $x_1, \dots, x_{2n}$  are approximate solutions that depend on the accuracy of  $\Psi = \text{constant}$ .

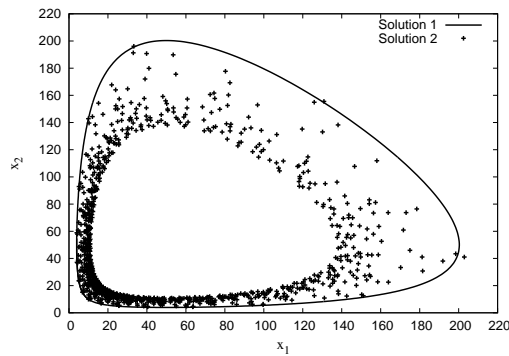


Figure 3.1: Numerical simulations of classical LV equation. Solution 1 and Solution 2 have different increment.

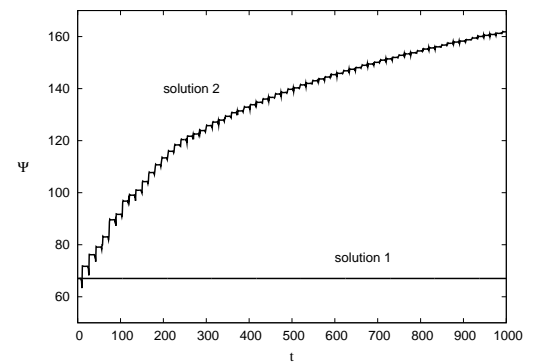


Figure 3.2: Numerical simulation of conserved quantity  $\Psi$ . Solution 1 are constant but Solution 2 oscillate and increase with respect to time.



## 4 Examples

The conservation law (3.7) in the BCF is a velocity-independent form. In the work of J. Fernández-Núñez, some functions are velocity-dependent and coupled with derivative terms such as  $\dot{x}_{2i} \log x_{2i-1}/x_{2i}$  and  $\dot{x}_{2i-1} \log x_{2i}/x_{2i-1}$  to obtain the classical LV system, and we also found a velocity-independent form of the exponential type,  $\exp\{f(x_1, x_2, \dots)\}$ , shown in the example in section 4.2.

### 4.1 2-variable system

Here, we show an example of  $2n$ -dimensional ND system in the case of  $n = 1$  in (3.1). The ND system of (3.1) is defined by

$$\begin{aligned}\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_1^2 + a_{14}x_1x_2, \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_1x_2 + a_{24}x_2^2.\end{aligned}\tag{4.1}$$

The mixing interactions and self-interactions are expressed as  $x_1x_2$ ,  $x_1^2$  and  $x_2^2$ . Since this system has two variables, we have only  $x_1x_2$  mixing interaction. The Lagrangian of this system is described from (3.2) as

$$\begin{aligned}\mathcal{L} &= \alpha_1\dot{x}_1x_2 + \alpha_2x_1\dot{x}_2 + \alpha_3x_1^2 + (\alpha_4 + \alpha_5)x_1x_2 \\ &\quad + \alpha_6x_2^2 + \alpha_7x_1^2x_2 + \alpha_8x_1x_2^2.\end{aligned}\tag{4.2}$$

From (4.2), we get the following nonlinear differential equation,

$$\dot{x}_1 = \frac{1}{d_{21}}\{(\alpha_4 + \alpha_5)x_1 + 2\alpha_6x_2 + 2\alpha_8x_1x_2 + \alpha_7x_1^2\},\tag{4.3}$$

$$\dot{x}_2 = \frac{1}{d_{12}}\{2\alpha_3x_1 + (\alpha_4 + \alpha_5)x_2 + 2\alpha_7x_1x_2 + \alpha_8x_2^2\}.\tag{4.4}$$

The parameter  $d_{21}$  is given by  $d_{21} = \alpha_2 - \alpha_1 = -d_{12}$ . The conserved quantity  $\Psi$  of this system is obtained as follows

$$\Psi \equiv \alpha_3x_1^2 + (\alpha_4 + \alpha_5)x_1x_2 + \alpha_6x_2^2 + \alpha_7x_1^2x_2 + \alpha_8x_1x_2^2.\tag{4.5}$$

We show numerical simulations of 2-variable ND system in figs.4.1, 4.2, 4.3, and 4.4. Note that 2-variable ND 1 in fig.4.1 and 2-variable ND 2 in fig.4.2 with different parameters are periodically stable and have the same conserved quantity  $\Psi$  defined as (4.5). The parameters of both 2-variable ND 1 and ND 2 are given in caption fig.4.1 and fig.4.2. The solutions of 2-variable ND 1 and ND 2 are periodic and steady state in fig.4.3, and the conserved quantity defined in (4.5) is given in fig.4.4. The conserved quantity of 2-variable ND 1 and ND 2 is strictly constant with respect to time.

In fig.4.5 and fig.4.6, we show an example in the case that parameters of  $2n$ -ND system (3.1) do not satisfy the condition (3.5) and (3.6). The conserved quantity  $\Psi$  is not equal to a constant in time. In the simulation of 2-variable ND 3\* in fig.4.5, the coefficient of  $x_1^2$  in (4.3) changed from  $\alpha_7$  to  $-\alpha_7$ , and the coefficient  $d_{12}$  in (4.4) changed from  $d_{12}$  to  $d_{21}$  and the coefficient  $x_2^2$  in (4.4) also changed from  $\alpha_8$  to  $-\alpha_8$  so that the system of differential equations does not satisfy the condition (3.5), (3.6) and the conservation law (4.5). The parameters are determined as  $\alpha_1 = 1.0$ ,  $\alpha_2 = 2.0$ ,  $\alpha_3 = 0.5$ ,  $\alpha_4 = 0.5$ ,  $\alpha_5 = 0.5$ ,  $\alpha_6 = 0.1$ ,  $\alpha_7 = 0.01$ ,  $\alpha_8 = 0.01$ , and initial values are  $x_1 = 10.0$ ,  $x_2 = 10.0$ . In the simulation 4.6, quantity  $\Psi$  increased exponentially and diverged.  $\Psi$  is not constant with respect to time in this case. It indicates that  $\Psi$  is not constant when the



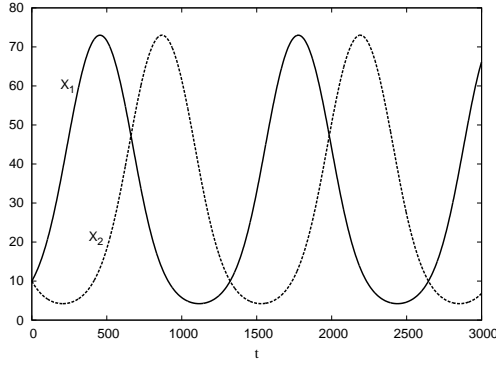


Figure 4.1: Numerical simulation of 2-variable ND 1. The parameters of 2-variable ND 1:  $\alpha_1 = 1.0$ ,  $\alpha_2 = 2.0$ ,  $\alpha_3 = -0.01$ ,  $\alpha_4 = 0.5$ ,  $\alpha_5 = 0.5$ ,  $\alpha_6 = -0.01$ ,  $\alpha_7 = -0.01$ ,  $\alpha_8 = -0.01$ , initial values are  $x_1 = 10.0$ ,  $x_2 = 10.0$ .

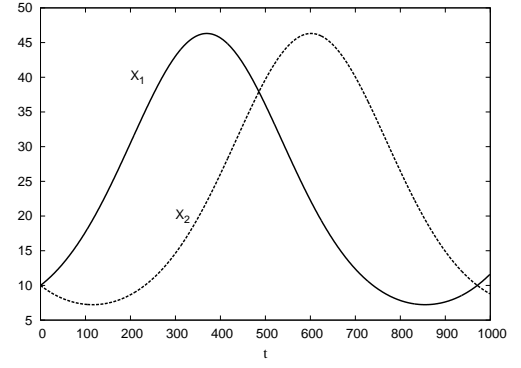


Figure 4.2: Numerical simulation of 2-variable ND 1. The parameters of 2-variable ND 2:  $\alpha_1 = 1.0$ ,  $\alpha_2 = 2.0$ ,  $\alpha_3 = -0.1$ ,  $\alpha_4 = 0.5$ ,  $\alpha_5 = 0.5$ ,  $\alpha_6 = -0.1$ ,  $\alpha_7 = -0.01$ ,  $\alpha_8 = -0.01$ , initial values are  $x_1 = 10.0$ ,  $x_2 = 10.0$ .

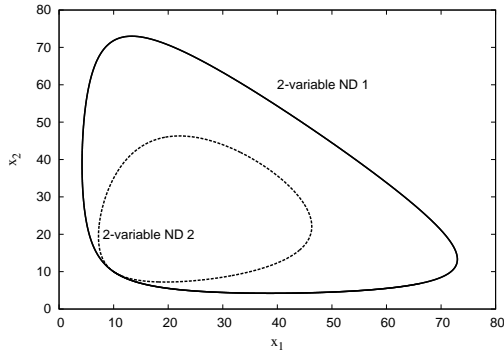


Figure 4.3: Numerical solutions of 2-variable ND 1 and ND 2.

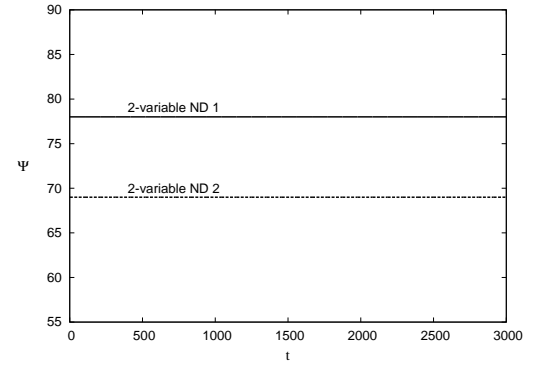


Figure 4.4: Numerical solutions of conserved quantity  $\Psi$  of 2-variable ND 1 and ND 2. Note that  $\Psi$  is constant in both cases.

coefficients do not meet the condition (3.5) and (3.6), and  $\Psi$  diverged when the solutions of the system diverged with respect to time.

Next, let us assume that the 3-variable system has a similar conservation law as (4.5), and we obtain the Lagrangian of the following type,

$$\begin{aligned} \mathcal{L} = & \alpha_1 \dot{x}_1 x_2 + \alpha_2 \dot{x}_2 x_3 + \alpha_3 \dot{x}_3 x_1 + \alpha_4 x_1 x_2 + \alpha_5 x_1 x_3 \\ & + \alpha_6 x_2 x_3 + \alpha_7 x_1^2 + \alpha_8 x_2^2 + \alpha_9 x_3^2 + \alpha_{10} x_1^2 x_2 \\ & + \alpha_{11} x_1 x_2^2 + \alpha_{12} x_1^2 x_3 + \alpha_{13} x_1 x_3^2 + \alpha_{14} x_2^2 x_3 + \alpha_{15} x_2 x_3^2 \end{aligned} \quad (4.6)$$

The conserved quantity of (4.6) is given by

$$\begin{aligned} \Psi = & \alpha_4 x_1 x_2 + \alpha_5 x_1 x_3 + \alpha_6 x_2 x_3 + \alpha_7 x_1^2 + \alpha_8 x_2^2 + \alpha_9 x_3^2 \\ & + \alpha_{10} x_1^2 x_2 + \alpha_{11} x_1 x_2^2 + \alpha_{12} x_1^2 x_3 + \alpha_{13} x_1 x_3^2 + \alpha_{14} x_2^2 x_3 + \alpha_{15} x_2 x_3^2 \end{aligned} \quad (4.7)$$

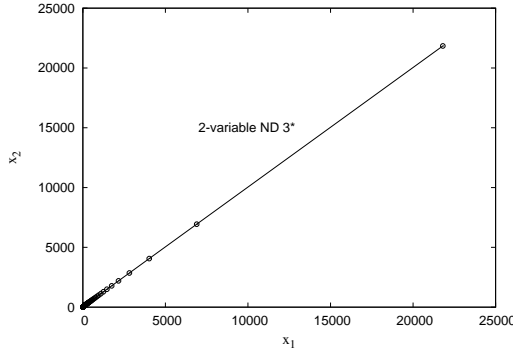


Figure 4.5: Numerical solutions of 2-variable ND 3\* system with parameters which do not satisfy conditions.

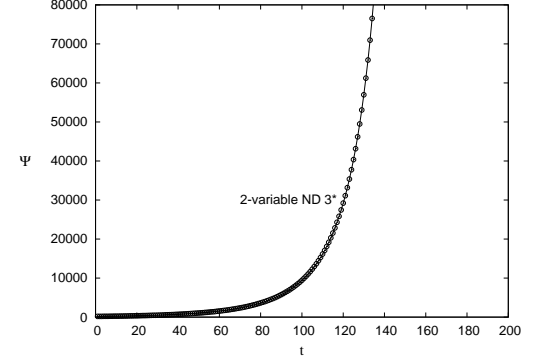


Figure 4.6: Numerical solutions of  $\Psi$  of 2-variable ND 3\*. Note that the solutions are not constant and diverge with respect to time.

The Lagrangian (4.6) produces, the following ordinary differential equations,

$$\begin{aligned} \alpha_1 \dot{x}_2 - \alpha_3 \dot{x}_3 &= 2\alpha_7 x_1 + \alpha_4 x_2 + \alpha_5 x_3 + 2\alpha_{10} x_1 x_2 + \alpha_{11} x_2^2 \\ &\quad + 2\alpha_{12} x_1 x_3 + \alpha_{13} x_3^2, \\ \alpha_2 \dot{x}_3 - \alpha_1 \dot{x}_1 &= \alpha_4 x_1 + 2\alpha_8 x_2 + \alpha_6 x_3 + 2\alpha_{11} x_1 x_2 + 2\alpha_{14} x_2 x_3 \\ &\quad + \alpha_{10} x_1^2 + \alpha_{15} x_3^2, \\ \alpha_3 \dot{x}_1 - \alpha_2 \dot{x}_2 &= \alpha_5 x_1 + \alpha_6 x_2 + 2\alpha_9 x_3 + \alpha_{12} x_1^2 + 2\alpha_{13} x_1 x_3 \\ &\quad + \alpha_{14} x_2^2 + 2\alpha_{15} x_2 x_3. \end{aligned} \quad (4.8)$$

The type of differential equations is a little different from that of the  $2n$  variables. The coupled linear equation (4.8) can be solved for  $x_3$ , resulting in

$$\begin{aligned} (\alpha_2 \alpha_{13} + \alpha_3 \alpha_{15}) x_3^2 &+ \{(2\alpha_1 \alpha_{15} + 2\alpha_3 \alpha_{14}) x_2 + (2\alpha_1 \alpha_{13} + 2\alpha_2 \alpha_{12}) x_1 \\ &\quad + (2\alpha_1 \alpha_9 + \alpha_2 \alpha_5 + \alpha_3 \alpha_6)\} x_3 + (\alpha_1 \alpha_5 + 2\alpha_2 \alpha_7 + \alpha_3 \alpha_4) x_1 \\ &\quad + (\alpha_1 \alpha_6 + \alpha_2 \alpha_4 + 2\alpha_3 \alpha_8) x_2 + (\alpha_1 \alpha_{12} + \alpha_3 \alpha_{10}) x_1^2 \\ &\quad + (2\alpha_3 \alpha_{11} + 2\alpha_2 \alpha_{10}) x_1 x_2 + (\alpha_1 \alpha_{14} + \alpha_2 \alpha_{11}) x_2^2 = 0. \end{aligned} \quad (4.9)$$

This is another time-independent relation of the 3-variable ND system. However, the time-independent relation is obtained from the differential equations, not from the conservation law  $\Psi$ . Hence, it indicates that there are two types of time-independent relations: (1) the time-independent relation that is strictly determined by Lagrangian or Noether's theorem; (2) the time-independent relation that is derived from a particular structure of differential equations, but is not necessarily related to the conservation law. The conserved quantity,  $\Psi$ , is strictly constant with respect to time, which is more general than the relation (4.9).

## 4.2 Exponential types of 2 variable coupled LV system

We show another type of conserved quantity of 2 variables which is similar to Lyapunov function. The classical type of LV equation is defined as

$$\begin{aligned} \dot{x}_1 &= a_{11} x_1 + a_{12} x_1 x_2, \\ \dot{x}_2 &= a_{21} x_2 + a_{22} x_1 x_2. \end{aligned} \quad (4.10)$$

$x_1$  and  $x_2$  are interpreted as a prey and a predator, respectively. The Lagrangian of (4.10) is described as follows

$$\mathcal{L} = \exp\{\alpha_1 x_1 + \alpha_2 x_2\} \{\alpha_3 \dot{x}_1 + \alpha_4 \dot{x}_2 + \alpha_5 x_1 x_2\}. \quad (4.11)$$

We can get the conditions of parameter from (4.10) and (4.11),

$$\begin{aligned} \dot{x}_1 &= \frac{1}{d_{12}} \alpha_5 \{\alpha_2 x_1 x_2 + x_1\}, \\ \dot{x}_2 &= \frac{1}{d_{21}} \alpha_5 \{\alpha_1 x_1 x_2 + x_2\}, \end{aligned} \quad (4.12)$$

where  $d_{12} = \alpha_1 \alpha_4 - \alpha_2 \alpha_3 = -d_{21}$ . This is the condition for parameters to have a conserved quantity. The conserved quantity  $\Psi$  is obtained as

$$\Psi \equiv \alpha_5 \exp\{\alpha_1 x_1 + \alpha_2 x_2\} x_1 x_2. \quad (4.13)$$

The logarithm of (4.13) is similar to a well known Lyapunov function  $V(x_1, x_2)$  which has the property  $dV(x_1, x_2)/dt = 0$ . It can be directly explained that the time derivative of conserved quantity should vanish,  $d\Psi/dt = 0$ , and therefore, Lyapunov function has a strong relation with Noether's theorem.

One can see that the conservation law and Lyapunov function are explicitly related to each other in the case of this simple example. However, it is not clear whether the connections between the  $2n$ -dimensional ND system and Lyapunov function are directly related to each other. Let us assume 3-variable type of differential equations of exponential type. The Lagrangian of linear 3-variable first-order equation is assumed as

$$\begin{aligned} \mathcal{L} = \exp\{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3\} \{ &\alpha_4 \dot{x}_1 + \alpha_5 \dot{x}_2 + \alpha_6 \dot{x}_3 + \alpha_7 x_1 + \alpha_8 x_2 \\ &+ \alpha_9 x_3 + \alpha_{10} x_1 x_2 + \alpha_{11} x_1 x_3 + \alpha_{12} x_2 x_3 \}. \end{aligned} \quad (4.14)$$

Applying Eq. (2.2) to Eq. (4.16), we get three equations as

$$\begin{aligned} &(\alpha_1 \alpha_5 - \alpha_2 \alpha_4) \dot{x}_2 + (\alpha_1 \alpha_6 - \alpha_3 \alpha_4) \dot{x}_3 + \alpha_1 \alpha_7 x_1 + (\alpha_1 \alpha_8 + \alpha_{10}) x_2 \\ &+ (\alpha_1 \alpha_9 + \alpha_{11}) x_3 + \alpha_1 \alpha_{10} x_1 x_2 + \alpha_1 \alpha_{11} x_1 x_3 + \alpha_1 \alpha_{12} x_2 x_3 + \alpha_7 = 0, \\ &(\alpha_2 \alpha_4 - \alpha_1 \alpha_5) \dot{x}_1 + (\alpha_2 \alpha_6 - \alpha_3 \alpha_5) \dot{x}_3 + (\alpha_2 \alpha_7 + \alpha_{10}) x_1 + \alpha_2 \alpha_8 x_2 \\ &+ (\alpha_2 \alpha_9 + \alpha_{12}) x_3 + \alpha_2 \alpha_{10} x_1 x_2 + \alpha_2 \alpha_{11} x_1 x_3 + \alpha_2 \alpha_{12} x_2 x_3 + \alpha_8 = 0, \\ &(\alpha_3 \alpha_4 - \alpha_1 \alpha_6) \dot{x}_1 + (\alpha_3 \alpha_5 - \alpha_2 \alpha_6) \dot{x}_2 + (\alpha_3 \alpha_7 + \alpha_{11}) x_1 + (\alpha_3 \alpha_8 + \alpha_{12}) x_2 \\ &+ \alpha_3 \alpha_9 x_3 + \alpha_3 \alpha_{10} x_1 x_2 + \alpha_3 \alpha_{11} x_1 x_3 + \alpha_3 \alpha_{12} x_2 x_3 + \alpha_9 = 0. \end{aligned} \quad (4.15)$$

The conserved quantity is velocity-independent and given by

$$\begin{aligned} \Psi = \exp\{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3\} \{ &\alpha_7 x_1 + \alpha_8 x_2 + \alpha_9 x_3 + \alpha_{10} x_1 x_2 \\ &+ \alpha_{11} x_1 x_3 + \alpha_{12} x_2 x_3 \}. \end{aligned} \quad (4.16)$$

The nonlinear equation (4.15) can be solved for  $x_3$ , resulting in

$$\begin{aligned} x_3 = \frac{1}{\sigma_1 + \sigma_2 x_1 + \sigma_3 x_2} [ &\{\phi_3 \alpha_1 \alpha_7 + \phi_1 (\alpha_2 \alpha_7 + \alpha_{10}) + \phi_2 (\alpha_3 \alpha_7 + \alpha_{11})\} x_1 \\ &+ \{\phi_3 (\alpha_1 \alpha_8 + \alpha_{10}) + \phi_1 \alpha_2 \alpha_8 + \alpha_2 (\alpha_3 \alpha_8 + \alpha_{12})\} x_2 \\ &+ \{\phi_3 \alpha_1 \alpha_{10} + \phi_1 \alpha_2 \alpha_{10} + \phi_2 \alpha_3 \alpha_{10}\} x_1 x_2 + \phi_3 \alpha_7 + \phi_1 \alpha_8 + \phi_2 \alpha_9 ] \end{aligned} \quad (4.17)$$

where  $\phi_1 = \alpha_3 \alpha_4 - \alpha_1 \alpha_6$ ,  $\phi_2 = \alpha_1 \alpha_5 - \alpha_2 \alpha_4$ ,  $\phi_3 = \alpha_2 \alpha_6 - \alpha_3 \alpha_5$ ,  $\sigma_1 = -\phi_3 (\alpha_1 \alpha_9 + \alpha_{11}) - \phi_1 (\alpha_2 \alpha_9 + \alpha_{12}) - \phi_2 \alpha_3 \alpha_9$ ,  $\sigma_2 = -\phi_3 \alpha_1 \alpha_{11} - \phi_1 \alpha_2 \alpha_{11} - \phi_2 \alpha_3 \alpha_{11}$ ,  $\sigma_3 = -\phi_3 \alpha_1 \alpha_{12} - \phi_1 \alpha_2 \alpha_{12} - \phi_2 \alpha_3 \alpha_{12}$ . As discussed in the example 4.1, the conserved quantity is strictly constant with respect to time, which is more general than equation (4.17).

## 5 Conclusion

In this paper, we have investigated a system of  $2n$ -dimensional, coupled first-order differential equations which contains self-interactions, and mixing interactions by using Noether's theorem. We discussed that the nonlinear differential equations with a conserved quantity,  $\Psi$ , calculated by Noether's theorem have converged stable solutions, and the coefficients of nonlinear interactions are strictly confined by the conservation law of the system. The conserving converged solutions are shown by a closed curve in  $(x_1, x_2)$ -coordinates for 2-dimensional case, and in general, it should be discussed by a closed hyper-surface in  $(x_1, x_2, \dots, x_{2n})$ -coordinates for  $2n$ -dimensional case.

Conventionally, the analysis of conserved quantity of solutions to a system of coupled differential equations is discussed with Lyapunov function [20]. There are two main types of Lyapunov functions that are strict Lyapunov and non-strict Lyapunov functions. Our conserved quantity would correspond to the strict Lyapunov function, due to the global property of Lagrangian approach. The theorem relates stability and conserved quantity analogous to conservation laws of energy and momentum in physics, hence it is helpful to understand a system of nonlinear differential equations in view of scientific or physical terms.

A system of symmetric nonlinear coupled first order differential equations has a conservation law and exists in the form of the  $2n$ -independent variables  $(x_1, x_2, \dots, x_{2n})$ . We termed the system of symmetric first-order differential equations composed of the  $2n$ -dimensional nonlinear interactions as the binary-coupled form (BCF). If a coupled nonlinear system exists in a form of symmetric first-order differential equations, which has a velocity-independent conserved quantity as discussed in the paper, the system tends to be composed of the binary-coupled form  $(2, 4, 6, \dots, 2n)$ . The predator-prey type system (2-coupled form), food-web system [21, 22], and gene regulation network system [23, 24] may be typical examples, and also the computer network seems to be expressed by  $2n$ -coupled form [25]. If a competitive system is the odd-variable type, the system of the first-order differential equations becomes little different as shown in the example in section 4.2.

When a binary-coupled system with a conservation law  $\Psi(x_1, \dots, x_{2n})$  and another system with  $\Psi(x_{2n+1}, x_{2n+2})$  interact with each other and result in constructing a new binary-coupled system, the system should have the conservation law in the form of  $\Psi(x_1, \dots, x_{2n}) + \Psi(x_{2n+1}, x_{2n+2}) \rightarrow \Psi(x_1, \dots, x_{2n+2})$  as in the form (3.7), even if they are nonlinearly interacting as (3.1). This may indicate an addition law for the same conserving ND systems. The addition law may be interpreted as the restoration or rehabilitation phenomena known in a large system of neural network or computer network when a small disordered device or part of a large network system is replaced by a normal device.

The conservation law is also used to check the accuracy of numerical solutions of nonlinear differential equations. The binary-coupled differential equations have interesting properties as shown in this paper, and the system of BCF  $(2, 4, 6, \dots, 2n)$  seems to be found in social, natural and biological sciences. If a system of BCF is naturally discovered in nature, the consequences discussed in BCF system would help understand structures, interactions, evolution mechanism of biological systems as well as social, economic and environmental systems.

We have investigated and discussed the property of  $2n$ -dimensional ND system in general. Although  $2n$ -dimensional ND system is discussed as a continuous system by employing differential equations, it is useful to examine their qualitative character of the classical LV system. However, most of biological phenomena may be mesoscopic, and a discrete system would fit actual phenomena more than a continuous system. In the future work, we would like to extend our method to discrete systems [26] and examine the connection between the pattern and time scale of biological

or mesoscopic phenomena and gene-related data sets. We understand that the conservation law  $\Psi$  depends sensitively on the values of nonlinear coefficients, and we also hope to investigate the relations between coefficients and stability of solutions of  $2n$ -ND systems.

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